Groups satisfying chain conditions on f-subnormal subgroups

Martyn R. Dixon[1], **Maria Ferrara**[2], **Marco Trombetti**[2]

[1] **mdixon@ua.edu** — University of Alabama; [2] **maria.ferrara3@unina.it**, **marco.trombetti@unina.it** — Università degli Studi di Napoli Federico II

Introduction

A subgroup H of a group G is called f-**subnormal** in G if there is a finite chain of subgroups

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

such that either $|H_{i+1} : H_i|$ is finite or H_i is normal in H_{i+1}, for $0 \leq i \leq n-1$.

Clearly every subnormal subgroup of G is f-subnormal in G; however if $G_i \simeq S_3$ for $i \geq 1$, then the subgroup H_1, generated by (12) in G_1, is f-subnormal in G, the direct product of the G_i, but H_1 is not subnormal in G.

This generalization of subnormality was introduced by R.E. Phillips [9] in 1972. Groups all of whose subgroups are f-subnormal were shown to be finite-by-soluble in [3] and further results were obtained in [4, 8] and [5].

Together with **M.R. Dixon** we have studied groups satisfying various chain conditions on f-subnormal subgroups. First we review some known chain conditions on subnormal subgroups.

Chain conditions for subnormal subgroups

A group G satisfies the *minimal condition on subnormal subgroups* (denoted by *min-sn*) if every non-empty set of subnormal subgroups of G has a minimal element or, equivalently, if every descending chain of subnormal subgroups of G terminates in finitely many steps. Groups satisfying the *maximal condition on subnormal subgroups* (denoted by *max-sn*) can be defined in an analogous manner. The classes of groups satisfying min-sn (respectively max-sn) have made a frequent appearances in the literature (see [11]).

Similarly a group G is said to have the *weak minimal condition on subnormal subgroups* (denoted by *min-∞-sn*) if every descending chain of subnormal subgroups of G

$$H_0 \geq H_1 \geq H_2 \geq \ldots$$

has the property that only finitely many of the indices $|H_i : H_{i+1}|$ are infinite; there being a corresponding *weak maximal condition on subnormal subgroups* also (denoted by *max-∞-sn*). Some interesting papers devoted to the conditions min-∞-sn and max-∞-sn include [6, 7, 13].

Furthermore, a group G is said to satisfy the *double chain condition on subnormal subgroups* if for every chain of subnormal subgroups of the form

$$(2) \qquad \ldots \leq G_{-n} \leq \ldots \leq G_{-1} \leq G_0 \leq G_1 \leq \ldots \leq G_n \leq \ldots$$

there exists an integer k such that $G_i = G_k$ for all $i \geq k$ or $G_i = G_k$ for all $i \leq k$. The structure of certain generalized soluble groups satisfying the double chain condition on subnormal subgroups was recently described in [2].

Finally, we say that G satisfies the *weak double chain condition on subnormal subgroups* (respectively *f-subnormal subroups*) if in every chain of the form (2) where G_i is subnormal in G (respectively f-subnormal) at most finitely many of indices $|G_{i+1} : G_i|$ are infinite.

Now, we move on to the chain conditions we took into account in our study.

Chain conditions for f-subnormal subgroups

A group G is said to satisfy the *maximal condition on f-subnormal subgroups* (*max-$f$sn*) if every ascending chain of f-subnormal subgroups terminates in finitely many steps; equivalently every non-empty set of f-subnormal subgroups of G has a maximal element. There are similar definitions for groups satisfying the properties named *min-$f$sn*, *max-∞-fsn* and *min-∞-fsn* and for groups satisfying the *(weak) double chain condition on f-subnormal subgroups*.

All our results on groups satisfying the above chain conditions on f-subnormal subgroups can be summarized in the following catch-all theorem.

Theorem 1.
Let G be a group. Then

- **i** *G satisfies* **min-fsn** *if and only if G satisfies* **min-sn**;
- **ii** *G satisfies* **max-fsn** *if and only if G satisfies* **max-sn**;
- **iii** *G satisfies* **min-∞-fsn** *if and only if G satisfies* **min-∞-sn**;
- **iv** *G satisfies* **max-∞-fsn** *if and only if G satisfies* **max-∞-sn**;
- **v** *G satisfies the* **double chain condition on subnormal subgroups** *if and only if G satisfies the* **double chain condition on f-subnormal subgroups**;
- **vi** *G satisfies the* **weak double chain condition on subnormal subgroups** *if and only if G satisfies the* **weak double chain condition on f-subnormal subgroups**.

Chain condition for (non-f-subnormal)-subgroups

Many authors have considered groups with the minimal (respectively maximal, weak minimal, weak maximal) condition on non-subnormal subgroups. Whilst it is clear that every chain condition on (non-subnormal)-subgroups implies the analogous condition for (non-f-subnormal)-subgroups the following easy example shows that the converse is always false.

Example 1.
Let $S = S_3$ and consider the direct sum

$$G = S \times \operatorname*{Dr}_{i \in I} G_i$$

where $i \in I = \{5, 7, 11, 13, \ldots\}$ and each G_i is the cyclic group of order i. Obviously every subgroup of G is f-subnormal and so G satisfies all the chain conditions on (non-f-subnormal)-subgroups. If we now take a non-subnormal subgroup of S, which we denote by H, then $H \times K$ is not subnormal in G, whenever K is any subgroup of direct product of the G_i. Therefore G has an infinite double chain of non-subnormal subgroups, which guarantees that G satisfies none of the chain conditions on (non-subnormal)-subgroups.

The f-Wielandt subgroup

The *Wielandt subgroup* $w(G)$ of a group G was introduced by H. Wielandt [15] in 1958 as the intersection of the normalizers of all subnormal subgroups of G. It is the analogous to the *kern* or *norm* $N(G)$ of a group G, defined to be the intersection of the normalizers of all the subgroups of G, and originally introduced by R. Baer [1]. It was shown in [14] that $N(G) \leq Z_2(G)$, the second centre of G, and that $G/C_G(N(G))$ is abelian.

We define the f-**Wielandt subgroup** of a group G, $\overline{w}(G)$, as the intersection of the normalizers of all f-subnormal subgroups of G. Clearly

$$Z(G) \leq N(G) \leq \overline{w}(G) \leq w(G),$$

where $Z(G)$ is the centre of G and the following example proves that $\overline{w}(G)$ can be strictly contained between $Z(G)$ and $w(G)$.

Example 2.
Let C be a cyclic group of order 7, let θ be an automorphism of order 3 of C and put B to be the locally dihedral 2-group, that is, $B = \langle x \rangle \ltimes A$ where $A = C_{2^\infty}$ and x is such that $a^x = a^{-1}$, for elements a of A, and $x^2 = 1$.
Let $\Theta = \langle \theta \rangle$, $D = \Theta \ltimes C$ and consider the group $G = B \times D$.
The norm of the group B coincides with the center of B, while

$$\overline{w}(B) = w(B) = B.$$

For the group D, instead, we have that

$$Z(D) = N(D) = \overline{w}(D) = \{1\},$$

while $w(D) = D$.
It follows that for the group G, $Z(G) = N(G) = Z(B)$, $\overline{w}(G) = B$ and $w(G) = G$, namely, the norm, the f-Wielandt subgroup and the Wielandt subgroup are three different subgroups of G.

It is easy to prove that in a residually finite group G the subgroup $\overline{w}(G)$ is contained in the second center $Z_2(G)$ of G and if it is torsion-free, then it is also abelian. Moreover, if we restrict our attention to polycyclic groups we can actually prove that the f-Wielandt coincides with the center.

Finally, it was shown in [10] and [12] that if G is group satisfying min-sn, then $|G : w(G)|$ is finite and we obtained an analogous result for the f-Wielandt subgroup as a consequence of the following theorem which strongly depends upon **Theorem 1**.

Theorem 2.
Let G be a group satisfying min-sn and suppose that H is f-subnormal in G. Then H has only finitely many conjugates in G.

Corollary.
Let G be a group satisfying the minimal condition on subnormal subgroups. Then $\overline{w}(G)$ has finite index in G.

References

[1] R. Baer, *Dier Kern, eine charakteristiche Untergruppe*, Compositio Math. **1** (1935), 254–283.

[2] M. Brescia and F. de Giovanni, *Groups satisfying the double chain condition on subnormal subgroups*, Ric. Mat. Math. **65** (2016), 255–261.

[3] C. Casolo and M. Mainardis, *Groups in which every subgroup is f-subnormal*, J. Group Theory **4** (2001), no. 3, 341–365.

[4] C. Casolo and M. Mainardis, *Groups with all subgroups f-subnormal*, Topics in infinite groups, Quad. Mat., vol. 8, Dept. Math., Seconda Univ. Napoli, Caserta 2001 , pp. 77–86.

[5] M.R. Dixon, M. Ferrara, and M. Trombetti, *Groups in which all subgroups of infinite rank have bounded near defect*, Preprint.

[6] L.A. Kurdachenko, *Groups satisfying weak minimality and maximality conditions for subnormal subgroups*, Mat. Zametki **29** (1981), no. 1, 19–30, 154, English transl. in Math. Notes Acad. Sciences USSR, **29** (1981), 11–16.

[7] D.H. Paek, *Chain conditions for subnormal subgroups of infinite order or index*, Comm. Algebra **29** (2001), no. 7, 3069–3081.

[8] J.C. Lennox, *On groups in which every subgroup is almost subnormal*, J. London Math. Soc. (2) **15** (1977), no. 2, 221–231.

[9] R.E. Phillips, *Some generalizatios of normal series in infinite groups*, J. Austral. Math. Soc. **14** (1972), 496–502.

[10] D.J.S. Robinson, *On the theory of subnormal subgroups*, Math. Z. **89** (1965), 30–51.

[11] D.J.S. Robinson, *Finiteness Conditions and Generalized Soluble Groups vols. 1 and 2*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, Heidelberg, New York, 1972, Band 62 and 63.

[12] J.E. Roseblade, *On certain subnormal coalition classes*, J. Algebra **1** (1964), 132–138.

[13] A. Russo, *On groups satisfying the maximal and the minimal conditions for subnormal subgroups of infinite order or index*, Bull. Korean Math. Soc. **47** (2010), no. 4, 687–691.

[14] E. Schenkman, *On the norm of a group*, Illinois J. Math. **4** (1960), 150–152.

[15] H. Wielandt, *Über den Normalisator der subnormalen Untergruppen*, Math. Z. **69** (1958), 463–465.

Ischia Group Theory 2018
19–23 March